Problem Sheet 1 For Supervision in Week 8.

1. ★ (i) If A and B are finite sets with $|A| = m \geq 2$ and $|B| = 2$, how many **surjections** are there from A to B?

   Hint: look at functions that are **not** surjections.

   (ii) If $|A| = m \geq 3$ and $|B| = 3$, how many surjections are there from A to B?

   For further results see PJE q.16, p.184.

2. In a card game you are dealt a hand of 13 cards from a normal playing deck of 52 cards.

   i) How many different hands are possible?

   ii) How many different hands will contain all four aces?

   iii) How many different hands will contain no hearts?

   iv) How many different hands will contain at least one spade?

3. Let A be a finite set with $|A| = n$. For each $0 \leq r \leq n$ give a bijection from $\mathcal{P}_r(A)$ to $\mathcal{P}_{n-r}(A)$.

   Hence show that

   $$\binom{n}{r} = \binom{n}{n-r}$$

   for all $0 \leq r \leq n$. (*Without* looking at the factorial form of the binomial number.)

4. Let A be a finite set with $|A| = n$. Describe $\bigcup_{r=0}^{n} \mathcal{P}_r(A)$.

   Hence evaluate

   $$\sum_{r=0}^{n} \binom{n}{r},$$

   without using the Binomial Theorem.

5. i. Using the Binomial Theorem, prove that

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} = 0.$$

ii. Use this result along with Question 4 to evaluate

$$\text{a)} \sum_{\substack{r=0 \\ r \text{ even}}}^{n} \binom{n}{r} \quad \text{and} \quad \text{b)} \sum_{\substack{r=0 \\ r \text{ odd}}}^{n} \binom{n}{r}.$$

6. Expand $(4x-3y)^5$.

7. Use the Binomial Theorem to calculate

$$\text{i)} \sum_{r=0}^{n} \frac{3^r 5^{n-r}}{r!\,(n-r)!} \quad \text{and} \quad \text{ii)} \sum_{r=0}^{n} 3^{2r} 5^{n-2r} \binom{n}{r}.$$

8. Find $x > 0$ that satisfy

$$\text{i)}\ x^2 = \sum_{r=0}^{4} 4^r \binom{4}{r} \quad \text{and} \quad \text{ii)}\ x^2 = \sum_{r=0}^{3} 3^r \binom{3}{r}.$$

9. What is the coefficient of $x^{99}y^{101}$ in $(2x+3y)^{200}$?

10. ★ Prove by induction that $n^5 - n$ is divisible by 5 for all $n \geq 1$.

(**Hint** In the inductive step you assume result is true for $n = k$. When you consider the $n = k + 1$ case apply the Binomial Theorem.)